Method of controlling chaos in laser equations

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A method of controlling chaotic to laminar flows in the Lorenz equations using fixed points dictated by minimizing the Lyapunov functional was proposed by Singer, Wang, and Bau [Phys. Rev. Lett. **66**, 1123 (1991)]. Using different fixed points, we find that the solutions in a chaotic regime can also be periodic. Since the laser equations are isomorphic to the Lorenz equations we use this method to control chaos when the laser is operated over the pump threshold. Furthermore, by solving the laser equations with an occasional proportional feedback mechanism, we recover the essential laser controlling features experimentally discovered by Roy, Murphy, Jr., Maier, Gills, and Hunt [Phys. Rev. Lett. **68**, 1259 (1992)].

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Since the development of the control algorithm of chaos proposed by Ott, Grebogi, and Yorke [1,2], several experiments have been performed where the devices are first driven into chaotic regimes, and then, using appropriate dynamical control techniques, subsequentially restored to order. For information relevant to this Brief Report, we will discuss only the experiment on convective chaos [3] and the controlling of chaos in lasers [4].

For the first experiment [3], Singer, Wang, and Bau heated a vertical circular tube from the bottom half-circle. At high heating wattage, the temperature difference between 9 o'clock and 3 o'clock (T_9-T_3) was found to be chaotic in time. Then, with the appropriate time-dependent heating control, they successfully brought the system back to a "laminar" regime. This success was in agreement with the earlier finding that the solutions of the Lorenz equations provide a good resemblence to the observed flow in the loop [5].

In the second experiment we consider here, Roy, *et al.* [4] used a diode-laser-pumped Nd-doped yttrium aluminum garnet (YAG) system that contains a potassium titanyl phosphate (KTP) doubling crystal in a cavity. For a given orientation between the YAG and KTP crystals, and when the laser is pumped to about three times the threshold, chaos was observed. Using a technique of occasional proportional feedback (OPF), they were able to control chaos and the intensity became periodic again, even at the high pump rate.

In the following, we discuss the controlling method of the convective system [3]. In addition to the classical Lorenz equations, the authors [6,7] control the heating rate with the controlling term $\epsilon\,\mathrm{sgn}(z-z_0)$, where ϵ is the control constant and z is the temperature difference between 12 o'clock and 6 o'clock ($T_{12}-T_6$).

The Lorenz equations with this control become

$$\frac{dx}{dt}=\mathrm{Pr}(y-x)\ ,$$
$$\frac{dy}{dt}=-xz-y\ , \tag{1}$$
$$\frac{dz}{dt}=-[R+\epsilon\,\mathrm{sgn}(z-z_0)]-z+xy\ ,$$

where Pr is the Prandtl number and R is the Rayleigh number. The variables x, y, and z correspond, respectively, to the average velocity in the loop, T_3-T_9, and $T_{12}-T_6$.

To repeat this calculation we use a stiff differential solver (LSODE, performed on a CRAY-YMP supercomputer. The parameters used in this calculation are $\mathrm{Pr}=10$ and $R=28$ ($b=1$ in this case). The time step Δt is 0.01. The chaotic behavior of $x(t)$ for the case $\epsilon=0$ is shown in Fig. 1(a). In all our calculations, we use 2^{12} integration time steps, except for Fig. 3(c). The corresponding power spectrum is shown in Fig. 1(b). The phase plot of $y(t)$ vs $x(t)$ in Fig. 1(c) shows a typical Lorenz attractor.

With $\epsilon=2.069\,999\,99$ and $z_0=-1$, the function $x(t)$ changes from chaos to laminar [Fig. 2(a)]. This value is a consequence of Lyapunov functional analysis of the Lorenz equations [3].

Interesting results occur when $z_0=+1$, a new fixed point. We find a periodic behavior of $x(t)$ with the control parameter $\epsilon=2.6$ [Fig. 2(b)]. In this case, the average value of $x_{\mathrm{av}}=-5$. Furthermore, when $z_0=+1$ and $\epsilon=2.999\,999\,999$, we also find a periodic solution with $x_{\mathrm{av}}=+5$ [Fig. 2(c)]. The solution diverges at $\epsilon=3$. Between $\epsilon=0$ and $\epsilon=2.999\,999\,999$ there are windows of periodic $x(t)$ sandwiched between chaotic states. With this new control mechanism, we are able to force the new attractor to reside within the "eyes" of the Lorenz attractor shown in Fig. 1(c) ($|x_{\mathrm{av}}|=5$). Thus the periodicity of the types shown in Figs. 2(b) and 2(c) is the consequence of this new phenomenon.

In our study, controlling chaos in the Lorenz equations in a sense is analogous to entering the parameter space of a new map, with the controlling term, for the Lorenz equations. The control parameter consists of a constant R and a feedback parameter $\epsilon\,\mathrm{sgn}(z-z_0)$.

The laser experiment [4] motivates us to interpret our new findings of periodicity of the chaotic regime of the Lorenz equations with the periodicity of the intensity of a laser. It was pointed out in 1975 by Haken [8] that the Lorenz equations and the laser equations are mathematically isomorphic. In the laser equations, we have the field strength E with decay time κ, the polarization P with decay time γ, and the atomic inversion D with decay time $\gamma_{\parallel}$. In properly chosen units and with our new control term, they become

$$\frac{dE}{dt}=\kappa(P-E) ,$$

$$\frac{dP}{dt}=-\gamma ED-\gamma P , \qquad (2)$$

$$\frac{dD}{dt}=\gamma_{\parallel}[(\Lambda+1)+\epsilon\,\mathrm{sgn}(D-D_0)]-\gamma_{\parallel}D+\gamma_{\parallel}EP .$$

With the proper identification of the parameters, the two equations can be shown to be equivalent. The inversion $D=\sum_\mu \sigma_\mu$ in our paper needs some elaboration. Here σ_μ is the difference of the occupational numbers N_1 and N_2

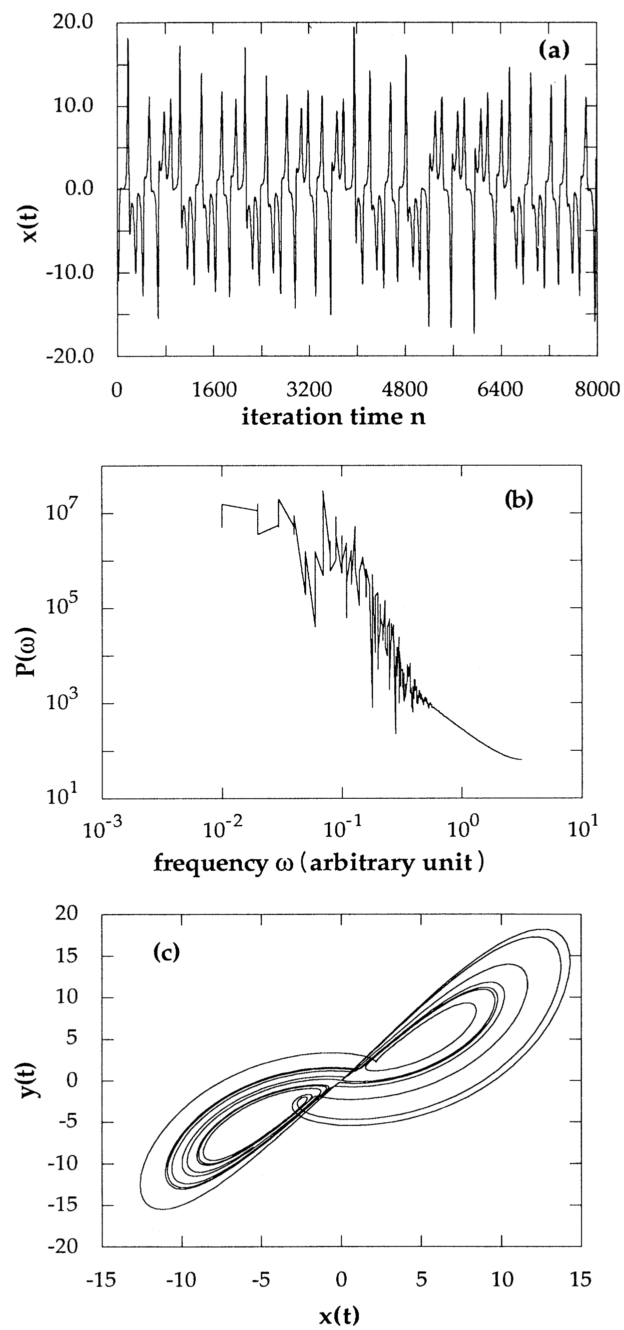

FIG. 1. (a) $x(t)$ of the Lorenz equations with Pr=10, $R=28$, ($b=1$). In all our studies, the time $t=0.01n$, where n is the integration time step. (b) The power spectrum $P(\omega)$ of the function $x(t)$ of (a). (c) Plot of $y(t)$ vs $x(t)$ of (a), showing a typical Lorenz attractor.

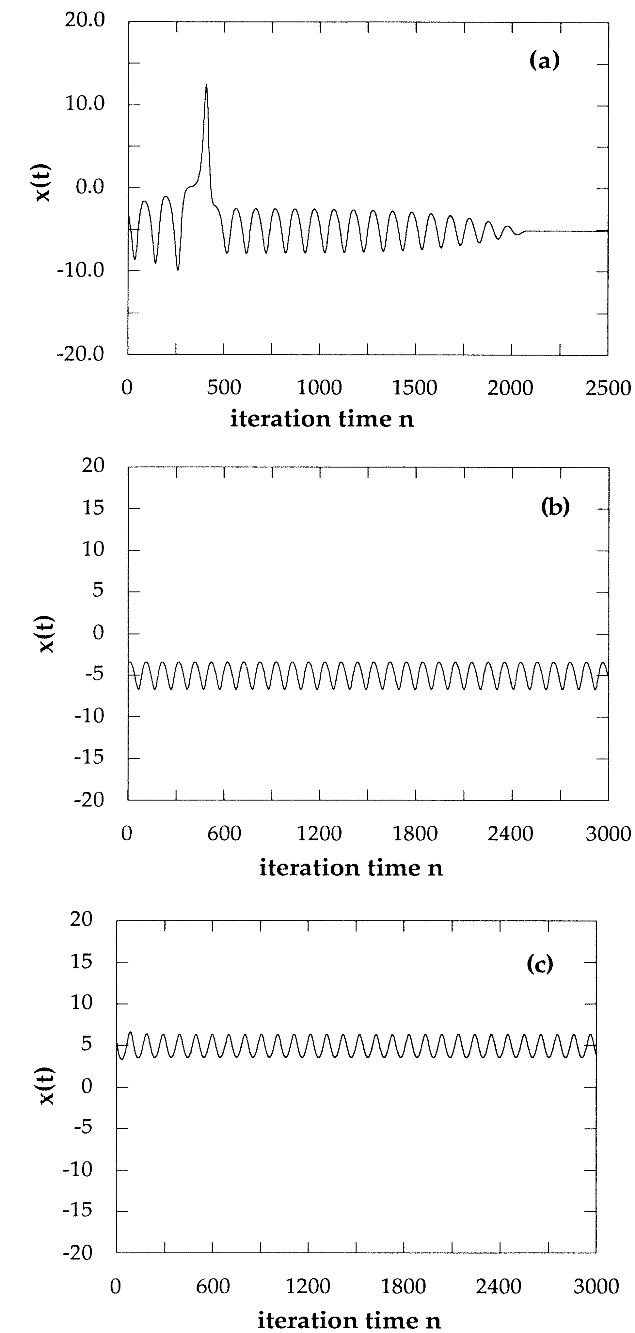

FIG. 2. (a) $x(t)$ of Eq. (1) with control $z_0=-1$ and $\epsilon=2.069\,999\,99$. (b) $x(t)$ of Eq. (1) with control $z_0=+1$ and $\epsilon=2.60$. (c) $x(t)$ of Eq. (1) with control $z_0=+1$ and $\epsilon=2.999\,999\,999\,9$.

of the lower and upper atomic energy levels of the atom μ, namely, $\sigma_\mu = N_1 - N_2$. This definition of σ_μ is chosen to be consistent with Eq. (1). Furthermore, with the following identification,

$$\begin{aligned} &t \to t\frac{\mathrm{Pr}}{\kappa} , \\ &E \to (R-1)^{-1/2}x , \\ &P \to (R-1)^{-1/2}y , \\ &D \to z , \\ &\Lambda = R-1 , \\ &\gamma = \gamma_{\parallel} = \frac{\kappa}{\mathrm{Pr}} , \end{aligned} \tag{3}$$

we recover the laser equations from the Lorenz equations. The validity of the analogy between the laser equations with the Lorenz equations has been extensively investigated by various workers [9–12].

For simplicity, we have used the Lorenz equations in the form of Ref. [3] with $b = \gamma_{\parallel}/\gamma = 1$. The controlling method using the parameter $\epsilon\,\mathrm{sgn}(z - z_0)$ in our study, i.e., with $z_0 = +1$, should be used as a controlling method for the laser equations. In the periodic windows, the average values of $x(t)$ are either $+5$ or -5. The intensities in these windows are positive since they are equal to the square of $x(t)$ or $E(t)$. The parameter

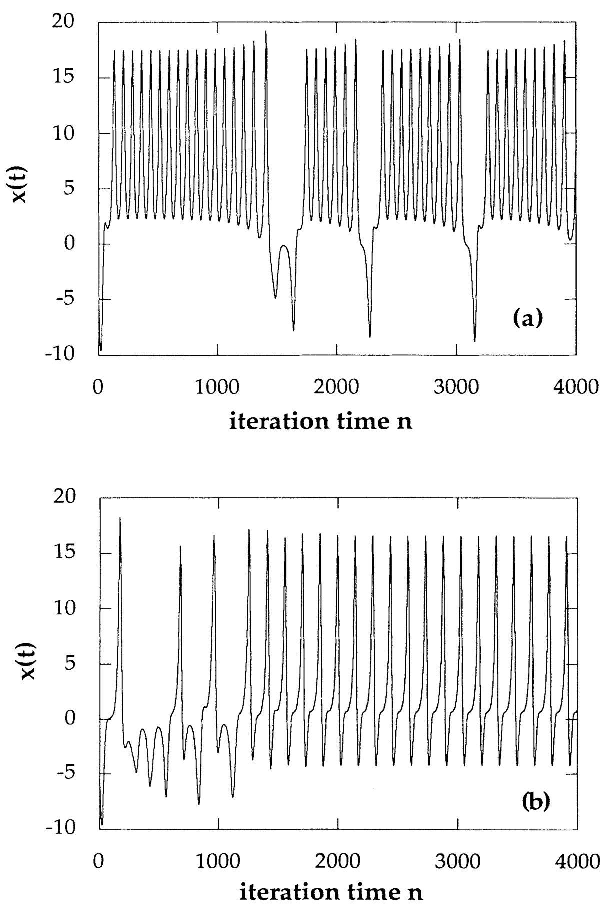

FIG. 3. (a) $x(t)$ of Eq. (5) with control $\lambda_c = 8.1512$ applied every 18 integration time steps for a duration of 15 time steps. (b) $x(t)$ of Eq. (5) with control $\lambda_c = 8.1512$ applied every 50 integration time steps for a durat6ion of 29 time steps.

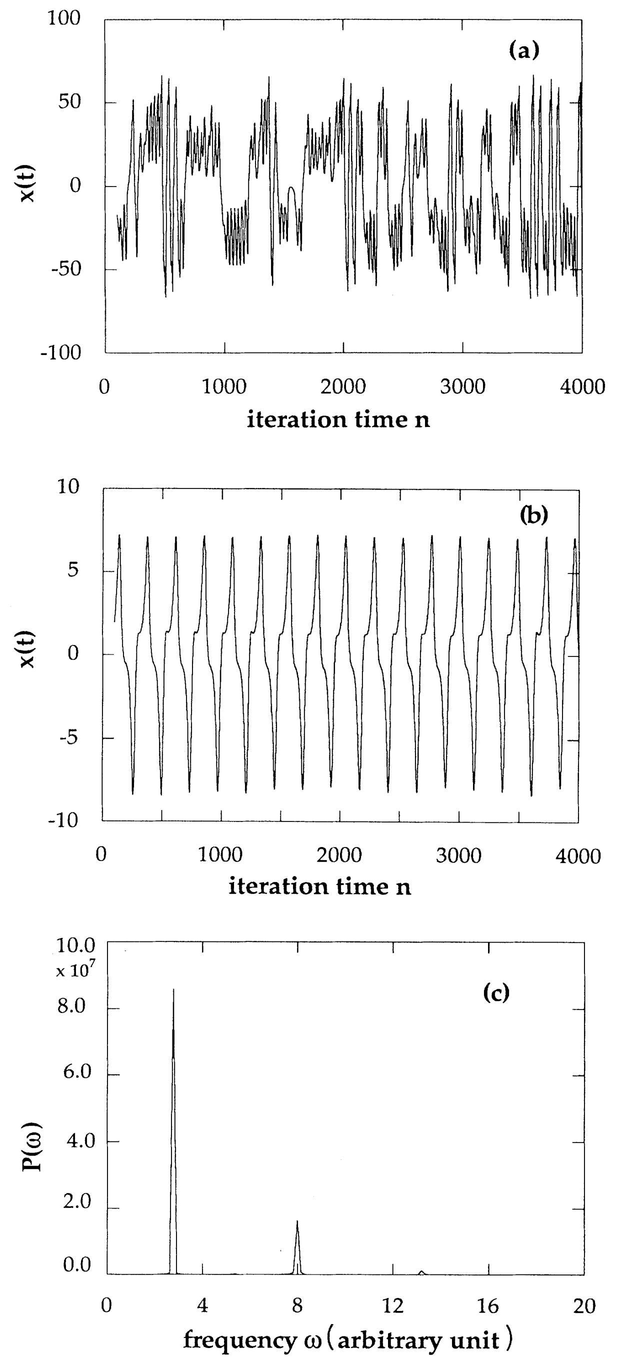

FIG. 4. (a) $x(t)$ of Eq. (5) modified with control λx^2 for $\lambda = 1.3$ applied every 30 integration time steps for a duration of 24 time steps. (b) $x(t)$ of Eq. (5) modified with control λx^2 for $\lambda = -1.0$ applied every 30 integration time steps for a duration of 24 time steps. (c) The power spectrum $P(\omega)$ of the function $x(t)$ of (b).

$$\Lambda=\frac{D_i-D_{\rm thr}}{D_{\rm thr}} \quad (4)$$

is the effective pump term of the laser, where D_i is the initial inversion and $D_{\rm thr}$ is the threshold inversion where instabilities start to occur. The results of our calculations for the Lorenz equations can be directly applied to the laser equations with the transformations in Eqs. (3) and (4). The control term, in this case, is $\epsilon\,\mathrm{sgn}(D-D_0)$. The inversion D, relative to a chosen $D_0=+1$, is used to determine the sign of the control term.

In the next section, we provide a control mechanism where the proportional feedback is different than that used in Eq. (1). This new feedback is influenced by the experimental work of Roy *et al.* [4]. Here, the intensity of the output of the laser is set in coincidence with a given periodic sequencer (OPF) and the resulting output of the intensity and the gate is used to add to the pump term. As a specific case, we use the proportional feedback scheme discussed below, and Eq. (1) becomes

$$\frac{dx}{dt}=\Pr(y-x)\ ,$$
$$\frac{dy}{dt}=-xz-y\ , \quad (5)$$
$$\frac{dz}{dt}=-\left[R+\lambda x\sum_{n}^{N}\pi(t-n\tau)\right]-z+xy\ ,$$

where λx is the proportional control term and the control is applied with frequency τ. In this study, $\pi(v)$ is a π-function defined as the difference of two Heaviside functions, $\pi(v)=H(v-a)-H(b-v)$, where $b-a$ is the time width of the control signal.

In the following we discuss the results of solving Eq. (5) with various controlling parameters, frequencies, and widths. The value of λ is set to 8.1512 and the proportional feedback is applied occasionally every 18 integration time steps, ($\tau=18\Delta t$), for a duration of 15 time steps ($b-a=15\Delta t$). The corresponding function $x(t)$ is shown in Fig. 3(a). When the integration time is doubled to 2^{13}, the function $x(t)$ changes somewhat but the fast frequency is still present. With much longer integration the function $x(t)$ settles to a fixed point.

With the same $\lambda_c=8.1512$, the feedback is applied every 50 integration time steps for a duration of 29 time steps. The result is shown in Fig. 3(b). The value of $\lambda_c=8.1512$ is found by solving Eq. (5); it is the value where we found that it takes the shortest number of iteration time steps to achieve periodicity. The periodicity of this plot sustains indefinitely. For any duration less than 20 time steps, the monochromatic nature of $x(t)$ is lost.

We change the control term to be proportional to λx^2 to mimic the fact that the intensity is the square of the E field. With $\lambda_c=1.3$ applied every 30 integration time steps for a duration of 24 time steps the signal is chaotic [Fig. 4(a)]. When $\lambda_c=-1.0$ is applied every 30 integration time steps for a duration of 24 time steps, the signal becomes periodic again [Fig. 4(b)] with the corresponding power spectrum shown in Fig. 4(c).

In general, with the various control methods discussed above, we can recover either fixed-point or periodic solutions of the Lorenz laser equations. The correct equations that describe the real laser system of Ref. [4] can be more complex than the Lorenz laser equations, but our study shows that the general features of the control of chaos in lasers can be modeled with the latter with some success.

The methods of controlling chaos have been rapidly developed [13–16]. In this Brief Report, we find that a laser operated at a chaotic mode can be brought back to a periodic mode with the various control mechanisms discussed above.

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